Shear Flow Aerodynamics: Lifting Surface Theory

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A lifting surface theory based on a parallel shear flow model is presented for steady, incompressible flows. The theory is intended to account approximately for the presence of a boundary layer. The method of Fourier transforms is used to calculate the pressure on a surface of infinite extent and arbitrary contour. Immediately above the surface is a region of sheared flow (the boundary layer), outside of which the flow velocity is constant. The Fourier transform of the pressure on this surface is used to derive the shear flow equivalent to the kernel function of classical potential flow lifting surface theory. The kernel function provides an integral relation between the upwash at a given point on the surface and the pressure everywhere on the surface. This relation is treated as an integral equation for the pressure, and is solved numerically. Computations are presented for the lift and pitching moment on a flat plate in two-dimensional flow, and for flat, rectangular wings of aspect ratio 1, 2, and 5. As expected, the shear layer decreases the lift curve slope; however, the shear layer (whose thickness is constant along the wing chord) has little effect on the center of pressure.

Nomenclature

Æ	= aspect ratio ($=2b/c$)
A_ν	= [see Eq. (19)]
b	= wing half-span
c	= wing chord
$f(x,y)$	= surface contour (Fig. 1)
$I_\nu(\ )$	= Bessel function of second kind and order ν
$J_{\bar{m}}(\ )$	= Bessel function of first kind and order m
$K(\ )$	= aerodynamic kernel function
$L(\ );L$	= [see Eq. (15)]; also wavelength of sinousoidal wall deflection
n	= exponent in shear layer velocity profile [(Eq. (5)]
p	= fluid pressure
$P_{\bar{m}}$	= pressure modal amplitude [Eqs. (23) and (23a)]
r	$=(x^2+y^2)^{1/2}$
R	$=(\alpha^2+\gamma^2)^{1/2}$
u,v,w	= fluid velocity components
U_I	= freestream velocity
$V(\ )$	= (see Appendix)
W	= upwash [Eq. (16)]
x,y,z	= coordinate axes (see Fig. 1)
α,γ	= Fourier transform variables
$\Gamma(\ )$	= gamma function
δ	= shear layer thickness
ν	$=1/2+1/n$
ρ	= fluid mass density
$\psi_m(\ ),\phi_n(\ )$	= modal pressure functions [Eqs. (23) and (23a)]
$(\ )^*$	= Fourier transform

I. Introduction

THE problem of calculating the lift and pitching moment on a wing of given shape is of central importance in aeronautics. There is a vast literature on the subject, known as lifting surface theory, which is extensively reviewed in Refs. 1 and 2. Common to all the works referred to is the assumption of potential flow. The presence of a boundary layer adjacent to the wing surface is ignored. In this paper we present a lifting surface theory for steady, incompressible flows based on a shear flow rather than a potential flow model. More specifically, instead of assuming that the initial flow velocity is constant, we specify that it varies with the coordinate normal to the wing surface, being zero on the surface itself, and increasing to a constant (freestream) value within a finite distance δ, from the wing. The velocity profile within this region of sheared flow is chosen to resemble that observed in viscous (turbulent) boundary layers. However, the thickness (δ) of the shear layer does not vary in the streamwise direction, as it actually would in a "real" boundary layer.

In Sec. II we calculate the Fourier transform of the pressure on an infinite surface of specified contour immersed in a shear layer. This result can also be interpreted directly as the pressure on a wavy wall of infinite extent. The essentially analytical solution is valid for arbitrary shear layer thicknesses unlike previous results,[3-6] which are based on the assumption that the thickness at the shear layer is small relative to the wavelength at the wall deformation. For a numerical approach see Dowell.[7]

In Sec. III the transformed pressure is employed to derive the kernel function that relates the upwash at any given point on the wing to the pressure distribution over the entire wing. The method by which this is accomplished was suggested in a previous paper by Dowell and Ventres.[8]

As noted above, the present work was motivated by a desire to include boundary-layer effects on lifting surface pressure distributions. It is well known that such effects are relatively more important for control surfaces than for the primary lifting surface. This is fortunate because the assumption used here of a constant boundary-layer thickness everywhere (on and off the airfoil) should be a better approximation for control surfaces than primary lifting surfaces. It is clearly desirable to extend the numerical computations to include control surface geometries. Of course, the present model may be thought of as the first term in a solution obtained by an expansion of boundary-layer thickness with respect to streamwise coordinate.† The first term in this expansion (i.e. the present solution) may be a better approximation than one might at first anticipate since, as we shall see, the effect of the

Received May 22, 1975; revision received February 20, 1975. This work was supported by NASA Grant 31-001-197, Ames Research Center. The author gratefully acknowledges many helpful discussions with E.H. Dowell.

Index category: Aircraft Aerodynamics.

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†Dowell has formally calculated the next term in this series (private communication).

shear layer on the aerodynamic kernel function is confined to a distance on the order of the shear layer thickness itself.

There is a substantial literature on a two-dimensional airfoil in a shear flow created by some external agent, e.g., atmospheric boundary layer or propeller slipstream. Tsien[9] and von Karman and Tsien[10] were among the earliest contributors, and the most recent work appears to be that of Ludwig and Erickson.[11] These authors also employed the assumption of a constant shear layer thickness. By using (piece-wise) linear shear profiles and restricting themselves to 2-dimensional flow, they were able to obtain solutions for airfoils of finite thickness and camber.

II. Shear Flow over a Surface of Specified Contour

Consider a shear flow over a surface whose deflection from the x-y plane is given as $z=f(x,y)$. In Fig. 1 the surface is shown as being finite in extent, but for the moment we shall assume that it covers the entire x-y plane. The surface deflection creates a small perturbation from an initially parallel shear flow $u=U(z)$, $v=w=0$. The function $U(z)$ is constant for $z>\delta$, so that the shear layer is limited to the region $0<z<\delta$ adjacent to the surface.

The momentum and continuity equations for the fluid are

$$uu_x+vu_y+wu_z+(1/\rho)p_x=0 \tag{1a}$$

$$uv_x+vv_y+wv_z+(1/\rho)p_y=0 \tag{1b}$$

$$uw_x+vw_y+ww_z+(1/\rho)p_z=0 \tag{1c}$$

$$u_x+v_y+w_z=0 \tag{1d}$$

Let u', v', w', and p' be the perturbation velocity components and the perturbation pressure. Then the total velocity and pressure are

$$u=U(z)+u' \tag{2a}$$

$$v=v' \tag{2b}$$

$$w=w' \tag{2c}$$

$$p=p_0+p' \tag{2d}$$

If these expressions are inserted in Eqs. (1), we obtain by retaining only linear terms in the perturbation quantities

$$Uu'_x+w'(\mathrm{d}U/\mathrm{d}z)+(1/\rho)p'_x=0$$

$$Uv'_x+(1/\rho)p'_y=0$$

$$Uw'_x+(1/\rho)p'_z=0$$

$$u'_x+v'_y+w'_z=0$$

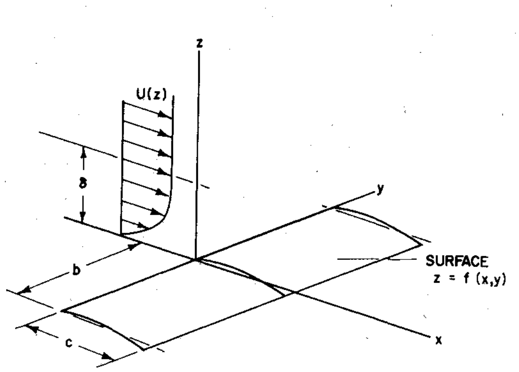

Fig. 1 Airfoil and flow geometry.

The perturbation velocity components u', v', and w' can be eliminated between these equations, producing a single equation for the perturbation pressure p (the prime is now dropped for convenience)

$$\nabla^2 p-(2/U)(\mathrm{d}U/\mathrm{d}z)p_z=0 \tag{3}$$

Let p^* be the double Fourier transform of p

$$p^*\equiv\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p\, e^{i(\alpha x+\gamma y)}\mathrm{d}x\mathrm{d}y$$

By applying the same transformation to Eq. (3), we see that the equation for p^* is

$$(\partial^2p^*/\partial z^2)-(2/U)(\mathrm{d}U/\mathrm{d}z)(\partial p^*/\partial z)-(\alpha^2+\gamma^2)p^*=0 \tag{4}$$

This is an ordinary linear differential equation. The variable coefficient in the second term vanishes for $z>\delta$, where the initial flow velocity is constant.

We now specify a $1/n$th power law for the shear flow velocity profile

$$U(z)\quad=U_1(z/\delta)^{1/n}\quad z<\delta \tag{5a}$$

$$=U_1\quad z>\delta \tag{5b}$$

With n fixed at a value between 7 and 11, this is an accepted approximation for turbulent boundary-layer flows at high Reynolds numbers. This expression does not hold, of course, within the very thin viscous sublayer immediately adjacent to the surface. Using this velocity profile, Eq. (4) becomes

$$\partial^2p^*/\partial z^2-(2/nz)(\partial p^*/\partial z)-R^2p^*=0 \tag{6a}$$

$$R\equiv(\alpha^2+\gamma^2)^{1/2} \tag{6b}$$

If we now impose a transformation of both the dependent and independent variables

$$p^*=\lambda^\nu g(\lambda) \tag{7a}$$

$$\lambda\equiv Rz \tag{7b}$$

and make the particular choice $\nu=1/2+1/n$, then Eq. (6) is recognized as a Bessel equation of the second kind of order ν

$$\mathrm{d}^2g/\mathrm{d}\lambda^2+1/\lambda(\mathrm{d}g/\mathrm{d}\lambda)-[(\nu^2/\lambda^2)+1]g=0 \tag{8}$$

The general solution of Eq. (8) is as follows

$$g=AI_\nu(Rz)+BI_{-\nu}(Rz) \tag{9}$$

The constants A and B are determined by boundary conditions imposed on the surface and at the outer edge of the shear layer. The boundary condition for the outer edge of the shear layer is obtained by noting that Eq. (4) reduces to $\partial^2p^*/\partial z^2-R^2p^*=0$ for $z>\delta$. Solutions to this equation have the forms $p^*=e^{+Rz},e^{-Rz}$. Of these, only the second is bounded at $z=+\infty$. Therefore, for $z>\delta$, p^* satisfies the equation

$$\partial p^*/\partial z+Rp^*=0 \tag{10}$$

We shall employ this relation as the boundary condition at $z=\delta$.

Strictly speaking, Eq. (6) is not correct for small z, since the velocity profile Eq. (5) does not apply within the viscous sublayer. On the other hand, the viscous sublayer is extremely thin at the Reynolds numbers pertaining to most aeronautical applications. It is reasonable, therefore, to accept the approximation that the thickness of the viscous sublayer is constant, and that the pressure variation across it can be neglected. The boundary condition that would normally be applied on the surface itself can then be transferred to the outer edge of the viscous sublayer, which we shall designate as $z=z_0$

$$\frac{\partial p^*}{\partial z}\bigg|_{z=z_0} = -\rho U \frac{\partial w}{\partial x}\bigg|_{z=z_0} \text{ where } w\bigg\|_{z=z_0} = U\bigg|_{z=z_0}\frac{\partial f}{\partial x} \tag{11a}$$

or

$$\frac{\partial p}{\partial z}\bigg|_{z=z_0} = -\rho U_1^2 (z_0/\delta)^{2/n} \cdot \frac{\partial^2 f}{\partial x^2} \tag{11b}$$

The Fourier transform of this boundary condition is

$$\frac{\partial p^*}{\partial z}\bigg|_{z=z_0} = -\rho U_1^2 (z_0/\delta)^{2/n} (i\alpha)^2 f^* \tag{12}$$

If we use Eqs. (10) and (12) to determine A and B from Eq. (9), and then let $z_0 \to 0$, we obtain the following result for the Fourier transform of the pressure on the surface

$$p^*/\rho U_1^2 = A^*(\alpha\delta,\nu\delta)\cdot W^*/U_1 \tag{13}$$

$$A^*(\alpha\delta,\gamma\delta) \equiv \frac{\Gamma(1+\nu)}{\nu\Gamma(1-\nu)} \; \frac{i\alpha}{R} \; \left(\frac{2}{\delta R}\right)^{2/n} L(\delta R) \tag{14}$$

$$L(\delta R) \equiv \frac{I_\nu(\delta R) + I_{\nu-1}(\delta R)}{I_{-\nu}(\delta R) + I_{1-\nu}(\delta R)} \tag{15}$$

In Eq. (13) we have introduced a new quantity W^*, the Fourier transform of the upwash that would exist on the plate if the shear layer were absent. That is

$$W^* \equiv [U_1(\partial f/\partial x)]^* = i\alpha U_1 f^* \tag{16}$$

This is done to facilitate comparisons with potential flow theory, i.e., when the shear layer thickness δ vanishes, or when the exponent $1/n$ in the velocity profile Eq. (5) goes to zero.

Equation (13) can be inverted (formally) using the convolution theorem to produce the pressure on a surface of infinite extent and arbitrary contour. Alternatively, one can calculate the pressure on a body of finite lateral extent and finite thickness, provided that it is symmetric about the $z=0$ plane. The flowfield is then also symmetric, so that the upwash off the body is zero. This situation is commonly referred to in the literature as the "thickness problem." Some care must be observed in computing the inverse of the function A^* defined in Eq. (14), since the integrals implied by the formal definition of the inversion formula are divergent. This problem can be circumvented in the manner described in Sec. III of this paper, and so the method will not be elaborated upon here.

Equation (13) can also be interpreted directly as the pressure (p^*) on a wavy wall of infinite extent. If we set $\gamma=0$ in Eq. (13), both the wall deformation and the flowfield become two dimensional, the wavelength of the wall deflection being $L=2\pi/\alpha$. Figure 2 shows the pressure on such a wall as a function of the shear layer thickness normalized by the deflection wavelength. The wall pressure drops off as the 2/nth power of the shear layer thickness. The pressure is directly out of phase with the wall deflection, that is, the pressure is greatest in the "valleys," just as it would be if the mean flow profile were uniform.

Also shown in Fig. 2 is an approximate expression for the wall pressure obtained by expanding Eqs. (13) through (15) in ascending powers of the normalized shear layer thickness $\alpha\delta$, retaining only the first two terms (the potential flow result and a first-order correction):

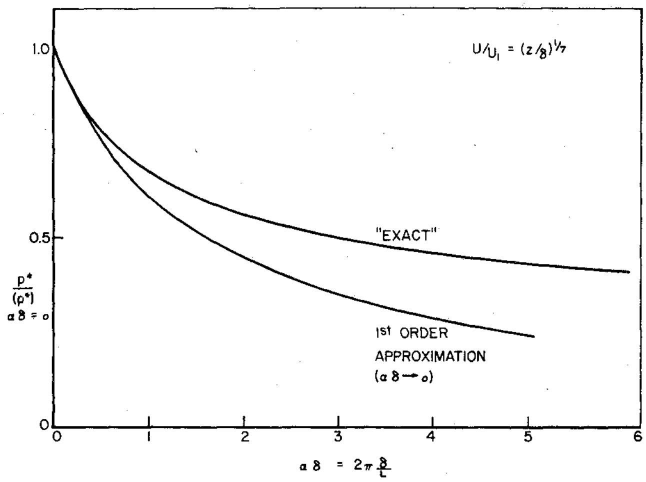

Fig. 2 Steady incompressible shear flow over a wavy wall.

$$p^*/\rho U_1^2 = (i\alpha/|\alpha|)\;(1-c_n|\delta\alpha|)\,W^* + 0(|\alpha\delta|^2) \tag{17a}$$

$$c_n \equiv 4n/(n+2)\;(n-2) \tag{17b}$$

or, alternately

$$p^*/\rho U_1^2 = (i\alpha/|\alpha|)\;W^*/(1+c_n|\alpha\delta|) + 0(|\alpha\delta|^2) \tag{17c}$$

Equations (17) can also be obtained from the perturbation analyses of Miles,[3] Ventres,[4] Yates,[5] and Lerner.[6] As has been noted by Yates, Eq. (17c) provides the best approximation to the exact solution. It is this form that is displayed in Fig. 2. Even Yates' form of the first-order approximation is substantially in error, however, for shear layers greater than half the wall wavelength in thickness.

III. Lifting Surface Theory

The foregoing results are not directly useful in calculating the lift on a wing or other lifting surface because the downwash off the wing is not known. The same problem is, of course, present in the theory of lifting surfaces in potential flow. The solution resorted to there is to assume a pressure distribution over the surface of the wing (the pressure is zero elsewhere in the $z=0$ plane) and to calculate the corresponding upwash distribution. The assumed pressure distribution is then adjusted in some manner so that the upwash distribution conforms to the prescribed shape of the wing. We shall use exactly the same procedure here to calculate the pressure distribution on a wing of zero thickness having a specified camber distribution and angle of attack. The pressure distribution on a wing having both thickness and camber is obtained by superimposing the solution for the symmetric thickness distribution (Sec. II), and the solution to be developed here for the pressure due to camber and/or angle of attack.

Equation (13) is rewritten to give W^* in terms of p^*

$$W^*/U_1 = K^*\,p^*/\rho U_1^2 \tag{18}$$

where K^* is simply the reciprocal of A^*. The inverse Fourier transform of this expression is a formal solution for the upwash anywhere in the $z=0$ plane (and specifically on the wing itself) in terms of the pressure distribution on the wing

$$\frac{W(x,y)}{U_1} = \int\!\!\int_{\text{wing}} K(x-\xi,y-\eta)\,\frac{p(\xi,\eta)}{\rho U_1^2}\,d\xi d\eta$$

In practice, it is not possible to invert K^* analytically, and (as we shall see) the most efficient and economical procedure for handling the calculation numerically does not involve an explicit computation of K. However, because of the fundamental importance of the kernel function in lifting surface theory, it is worthwhile to calculate K at least for two-dimensional flows, and to compare it to the equivalent kernel function of potential flow theory.

The Fourier inversion requires an integration over an infinite domain in alpha space

$$K(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} K^*(\alpha\delta) e^{i\alpha x} d\alpha$$

However, this integration over real values of alpha from $-\infty$ to $+\infty$ will not converge because K^* does not have the proper asymptotic behavior for large $|\alpha|$. The two-dimensional form of K^* is obtained from Eqs. (13) through (15) by setting $R = |\alpha|$. Since the limiting asymptotic form of the Bessel function $I_\nu(x)$ is independent of its index ν, the function L defined in Eq. (15) goes to 1 as $|\alpha\delta| \to \infty$. Therefore we have

$$K^* \sim -i\, A_\nu (|\alpha\delta|/2)^{2/n} \qquad (19)$$

where

$$A_\nu \equiv [\nu\Gamma(1-\nu)/\Gamma(1.\nu)]$$

as $|\alpha\delta| \to \infty$.

It is possible to break K^* up into two parts, one of which dies off rapidly as $|\alpha| \to \infty$, and so can be inverted numerically. The other can be handled analytically, using a suitable deformation of the path of integration in the complex alpha plane to circumvent the convergence problem. The two components of K^* are given in the following

$$K^* = K_1^* + K_2^*$$

where

$$K_1^* \equiv A_\nu (|\alpha|/i\alpha)\, (|\alpha\delta|/2)^{2/n} [1/L(|\alpha\delta|) - 1] \qquad (20a)$$

$$K_2^* \equiv A_\nu (|\alpha|/i\alpha)\, (|\alpha\delta|/2)^{2/n} \qquad (20b)$$

The numerical inversion of K_1^* is routine, since it dies off exponentially with $\alpha\delta$. The analytical inversion of K_2^* is described in the Appendix. The result is

$$K_2 = (A_\nu/\pi x)\, (\delta/2|x|)^{2/n} \Gamma(2\nu) \cos \pi/n \qquad (21)$$

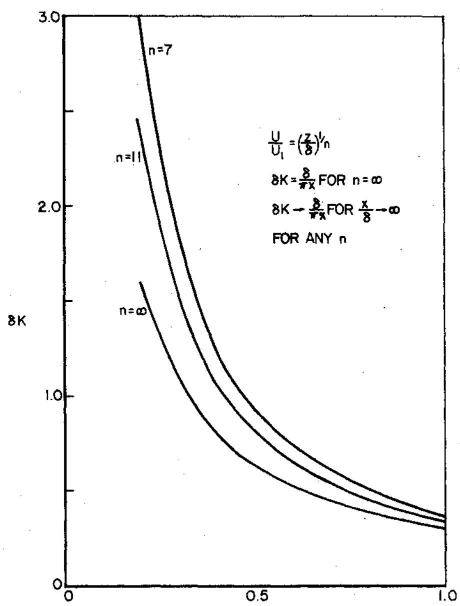

Fig. 3 Kernel function.

Figure 3 shows δK as a function of x/δ for $n = 7$ and 11. Also shown is the potential flow result

$$K = 1/\pi x \qquad (22)$$

which is labeled in the figure as $n = \infty$, since K takes on the potential flow value as n becomes large. All three curves have a common asymptotic limit as $x/\delta \to \infty$, and in fact are essentially identical for $x/\delta > 2$ or so. Since K may be interpreted physically as the downwash caused by a delta function distribution of pressure, this means that the influence of the shear layer is limited to a distance on each side of the source point comparable to the shear layer thickness. This suggests the possibility of accounting approximately for the effects of a varying boundary-layer thickness by inserting a variable $\delta(x)$ directly into the present theory, which has been developed for constant thickness shear layers.

To calculate the pressure distribution (and hence the total lift and pitching moment) on a wing of arbitrary contour, we express the pressure p as a linear combination of selected modal functions. For a two-dimensional flow (that is, a thin airfoil in a shear flow), we have

$$\frac{p}{\rho U_1^2} = \sum_{m=1}^{N} P_m \psi_m(x/c) \qquad (23)$$

The corresponding expression for three-dimensional flows includes a double summation and a second set of modal functions $\phi_n(y/b)$. For the moment we will concentrate on the two-dimensional theory, and list the formulas for three-dimensional flows later on.

We calculate the Fourier transform of Eq. (23) and insert it into Eq. (18). Carrying out the inverse transformation, we obtain

$$\frac{W}{U_1} = \sum_{m=1}^{N} P_m W_m(x/c) \qquad (24)$$

where W_m is the upwash associated with the pressure distribution ψ_m, and ψ_m^* is the Fourier transform of ψ_m

$$W_m(x/c) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} K^*(\bar{\alpha} \frac{\delta}{c}) \psi_m^*(\bar{\alpha}) e^{i\bar{\alpha}x/c} d\bar{\alpha} \qquad (25a)$$

$$\psi_m(\bar{\alpha}) \equiv \int_0^1 \psi_m(\xi) e^{-i\bar{\alpha}\xi} d\xi \qquad (25b)$$

$$\bar{\alpha} \equiv \alpha\, c \qquad (25c)$$

Again, because of the asymptotic behavior of K^*, this integral is only slowly convergent if computed numerically along the real alpha axis. (The corresponding expression in the three-dimensional theory involves an integration over an infinite transform plane, and is divergent.) It is more efficient, therefore, to resort to the same separation of K^* previously used. the model upwash functions $W_m(x/c)$ are thereby divided into two parts as well, one of which is computed by straightforward numerical inversion of the Fourier transform, as in Eqs. (25). The second part is computed by using the convolution theorem to invert the Fourier transform, followed by a subsequent integration over the chord of the airfoil

$$W_m^{(1)} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} K_1^*(\bar{\alpha} \frac{\delta}{c}) \psi_m^*(\bar{\alpha}) e^{i\alpha x/c} d\bar{\alpha} \qquad (26a)$$

$$W_m^{(2)} = \int_0^1 \psi_m(\xi) K_2 \left[\frac{x}{c} - \xi \right] d\xi \qquad (26b)$$

The line drawn through the integration symbol in the second of these expressions indicates that the Cauchy principal value of this integral is required. It will be noted from Eq. (21) that K_2 has a somewhat stronger singularity at the source point than does the potential flow kernel [Eq. (22)]. The appropriate definition of the Cauchy principal value is the same, however, in each case

$$-\!\!\!\!\!\!\int_{-a}^{+a} \frac{F(x)\,dx}{x\,|x|^{2/n}} \equiv \lim_{\epsilon\to 0^+}\left[\int_{-a}^{-\epsilon}\ldots dx+\int_{+\epsilon}^{+a}\ldots dx\right]$$

Once the modal upwash functions W_m have been computed numerically, Eq. (24) is transformed into a set of N equations in the N unknowns P_m by the technique known as collocation. That is, the upwash $W(x/c)$ is forced to conform to the specified upwash on the airfoil at N distinct points along the chord

$$W(x_i/c)=\sum_{m=1}^{N} P_m W_m(x_i/c)\quad i=1,2,\ldots N \tag{27}$$

These equations serve to determine the p_m uniquely. Given the p_m, the pressure distribution is calculated from the original modal Eq. (23) for p.

The modal functions $\psi_m(x/c)$ selected are those used in the theory of thin airfoils in a uniform potential flow. They are defined parametrically as follows

$$\psi_1(x/c)\equiv\cot\theta/2 \tag{28a}$$

$$\psi_m(x/c)\equiv\sin\,(m-1)\theta\quad m=2,3,\ldots N \tag{28b}$$

$$\cos\theta\equiv 1-2\,x/c \tag{28c}$$

Note that the Kutta condition is satisfied by each ψ_m, i.e.

$$\psi_m(1)=0$$

The first of these functions contains an $x^{-1/2}$ singularity at the leading edge, familiar from potential flow theory. It has not been shown that this is the appropriate singularity when a shear layer is present. However, the detailed nature of the singularity, if any, is probably not too important. In any case, the present theory does not present an accurate picture of the viscous flow near the leading edge, since on an isolated airfoil the boundary layer originates at the leading edge, and so is of vanishing thickness there.

The equivalent calculations for a rectangular wing in three-dimensional flow go through in an analogous manner. The steps corresponding to Eq. (23) through (27) are listed in the following

$$\frac{p}{\rho U_1^2}=\sum_m\sum_n P_{mn}\psi_m(x/c)\,\phi_n(y/b) \tag{23a}$$

$$\frac{W}{U_1}=\sum_m\sum_n P_{mn}W_{mn}(x/c,y/b) \tag{24a}$$

$$W_{mn}=W_{mn}{}^{(1)}+W_{mn}{}^{(2)}$$

$$W_{mn}{}^{(1)}=\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}K_1\left(\frac{\bar\alpha\delta}{c},\frac{\bar\gamma\delta}{c}\right)$$

$$\cdot\,\psi_m^*(\bar\alpha)\,\phi_n^*(\bar\gamma)\,e^{i(\bar\alpha x/c+\bar\gamma y/b)}\,d\bar\alpha\,d\bar\gamma$$

$$\phi_n^*(\bar\gamma)\equiv\int_{-1}^{+1}\phi_n(\eta)\,e^{-i\eta\bar\gamma}\,d\eta$$

$$\psi_m^*(\bar\alpha)\equiv\int_0^1\psi_m(\xi)\,e^{-i\xi\bar\alpha}\,d\xi$$

$$W_{mn}{}^{(2)}=\times\!\!\!\!\!\!\int_{-1}^{+1}\phi_n(\eta)\int_0^1\psi_m(\xi)$$

$$\cdot\,K_2[(x/c)-\xi,(y/b)-\eta]\,d\xi\,d\eta$$

$$K_1^*=A_2(R/i\alpha)\,(\delta R/2)^{2/n}[1/L(\delta R)-1]$$

$$\bar\alpha\equiv\alpha c\qquad\bar\gamma\equiv\gamma b$$

$$K_2(\xi,\eta)\equiv-(4\nu^2/\pi A\!\!\!R\,|\eta|^2)\,[(\delta/c)\,(2/A\!\!\!R\,|\eta|)]^{2/n}$$

$$\cdot\,V(2\xi/A\!\!\!R\,|\eta|)$$

$$V(x)\equiv\int_{-\infty}^{x}\frac{du}{(u^2+1)^{3/2+1/n}}$$

$$\frac{W(x_i/c,y_i/b)}{U_1}=\sum_m\sum_n P_{mn}\cdot\psi_m(x_i/c)\,\phi_n(y_i/b)$$

The X on the integration sign in the expression for $W_{mn}{}^{(2)}$ denotes the finite part of that integral, defined for functions with a singularity of the type displayed by K_2 as

$$\times\!\!\!\!\!\!\int_{-a}^{+a}\frac{F(y)\,dy}{y^{2+2/n}}\equiv\lim_{\epsilon\to 0}\left[\int_{-a}^{-\epsilon}\ldots dy\right.$$

$$\left.+\int_{+\epsilon}^{+a}\ldots dy-\frac{2F(0)}{(1+2/n)\,\epsilon^{1+2/n}}\right]$$

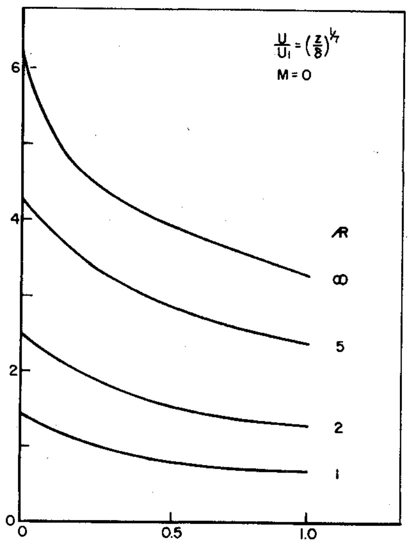

Fig. 4 Lift curve slope.

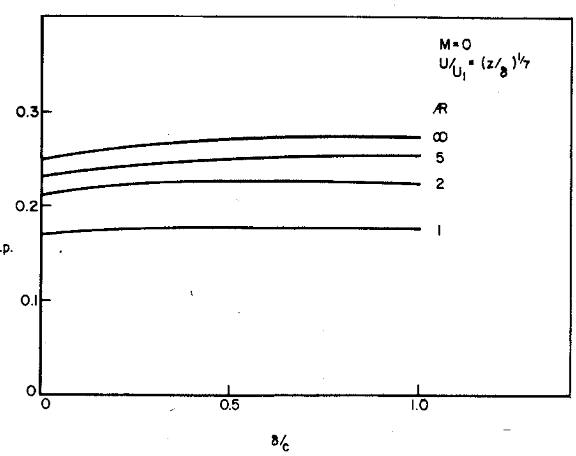

Fig. 5 Center of pressure.

The spanwise modal functions $\phi_n(y/b)$ are defined as follows

$$\phi_n(y/b) \equiv \sin(2n-1)\theta$$

$$\cos\theta \equiv y/b \qquad -1 \leq y/b \leq 1$$

These same functions are used in the classical lifting line theory of high aspect ratio wings in potential flow.

Because of the singularities present in the partial kernels K_2, some care must be excercised in carrying out the integration over the airfoil chord in the two-dimensional theory, and over the wing span in the three-dimensional theory. The latter spanwise integration is especially sensitive. The method used here for handling the integration in the vicinity of the singular point was borrowed from Ref. 12.

The foregoing theory has been used to compute the effect of a shear layer on the total lift and pitching moment of a flat plate, two-dimensional airfoil, and of a series of flat plate, rectangular wings of aspect ratio 1, 2, and 5. Both the airfoil and the wings have zero camber. Three chordwise modes were used in the two-dimensional calculation, while three modes in each of the spanwise and chordwise directions were used in the three-dimensional case. Check calculations made with seven modes for the two-dimensional problem demonstrated excellent covergence.

The effect of the shear layer on the slope of the lift curve for rectangular wings of aspect ratio 1,2, and 5 is shown in Fig. 4. Also shown is the lift curve slope for the two-dimensional airfoil, labeled $AR = \infty$. The shear layer decreases the slope of the lift curve, as expected.

Figure 5 shows the center of pressure travel for the same wings and airfoil. With no camber the center of pressure and the aerodynamic center coincide. The shear layer has relatively little effect on the center of pressure. Pressure distributions have also been calculated, but they will not be shown here because they are qualitatively similar to potential flow results. Roughly speaking, the shear layer diminishes the pressure at each point on the wing by an amount proportional to the decrease in lift curve slope.

IV. Conclusions

We have presented a means of computing the pressure distribution on lifting surfaces in incompressible shear flows whose velocity profile is $u/U_1 = (z/\delta)^{1/n}$. This profile is a good approximation for that found in viscous boundary layers that occur in the flow of real fluids at high Reynolds numbers. The numerical results presented indicate that the shear layer decreases the lift curve slope; however, the shear layer has but a small effect on the center of pressure.

The obvious improvement to be made on the theory presented herein is to allow for a variable shear layer thickness along the chord of the wing, and to eliminate the shear layer ahead of the wing. This complicates the fluid equations of motion considerably, since at least two variable flow components must then be included (if the streamwise velocity varies along the chord, the component of the velocity normal to the surface must also be nonzero to satisfy the continuity equation). However, it should be possible to make some approximations based on the "slowly" varying nature of the flow to simplify the situation somewhat. As noted in Sec. III, the shear layer kernel function differs from the potential flow version only for a distance along the chord comparable to the shear layer thickness δ. If δ itself varies but slightly in this interval, the *local* effect of the shear layer thickness variation should be small. The *global* effect on the other hand, could be included by utilizing an analysis based on a constant shear layer thickness, as described herein, and then including the variable boundary-layer thickness in the final stage of the computation.

V. Appendix

The formal definition of the Fourier inverse of K_2^* for two-dimensional flow is not convergent as it stands

$$K_2 = \frac{A_\nu}{2\pi} \int_{-\infty}^{+\infty} \frac{|\alpha|}{i\alpha} \left(\frac{|\alpha\delta|}{2} \right)^{2/n} e^{i\alpha x} \mathrm{d}\alpha \tag{1A}$$

since the integrand does not diminish in magnitude as $|\alpha| \to \infty$. However, from symmetry considerations K_2 can be reduced to

$$K_2 = \frac{A_\nu}{\pi} \int_0^\infty \left(\frac{\alpha\delta}{2} \right)^{2/n} \sin\alpha x \mathrm{d}\alpha \tag{2A}$$

This expression is no more convergent than the first, but it can be written as the imaginary part of the integral

$$\frac{A_\nu}{\pi} \int_0^\infty \left(\frac{\alpha\delta}{2} \right)^{2/n} e^{i\alpha x} \mathrm{d}\alpha \tag{3A}$$

which does converge if $Im(x) > 0$. We thus interpret K_2 as

$$K_2 = Im\left\{ \lim_{\epsilon \to 0} \frac{A_\nu}{\pi} \int_0^\infty \left(\frac{\alpha\delta}{2} \right)^{2/n} e^{i\alpha(x+i\epsilon)} \mathrm{d}\alpha \right\} \tag{4A}$$

This can be transformed into recognizable form by introducing

$$u \equiv \alpha[\epsilon - ix]$$

We obtain

$$K_2 = Im \frac{A_\nu}{\pi} \left(\frac{\delta}{2} \right)^{2/n} \left(\frac{i}{x+i\epsilon} \right)^{2/n+1} \int_0^\infty u^{2/n} e^{-u} \mathrm{d}u$$

The integral is the gamma function

$$\int_0^\infty u^{2/n} e^{-u} \mathrm{d}u = \Gamma(2/n+1)$$

$$= \Gamma(2\nu)$$

since $\nu = 4 + 1/n$. Taking the imaginary part, and letting $\epsilon \to 0$, we arrive at the final result:

$$K_2 = (A_\nu/\pi x)\ (\delta/2x)^{2/n} \Gamma(2\nu) \cos \pi/n \tag{5A}$$

The corresponding function for three-dimensional flow is

$$K_2 = \frac{A_\nu}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{R}{i\alpha} \left(\frac{\delta R}{2} \right)^{2/n} e^{i(\alpha x + \gamma y)} \mathrm{d}\alpha \mathrm{d}\gamma \tag{6A}$$

If we differentiate K_2 with respect to x the integrand is simplified so that it depends only on $R \equiv (\alpha^2 + \gamma^2)^{1/2}$. We then introduce the dual polar coordinate transformation

$$\alpha = R\cos\phi \qquad x = r\cos\theta$$

$$\gamma = R\sin\phi \qquad y = r\sin\theta \tag{7A}$$

so that

$$\frac{\partial K_2}{\partial x} = \frac{A_\nu}{4\pi^2} \int_0^\infty R^2 \left(\frac{\delta R}{2} \right)^{2/n} \times \int_0^{2\pi} e^{iRr\cos(\theta-\phi)} \mathrm{d}\phi \mathrm{d}R$$

The integral with respect to ϕ is

$$\int_0^{2\pi} e^{iRr\cos(\theta-\phi)} \mathrm{d}\phi = 2\pi J_0(Rr)$$

We thus obtain a relatively simple (but unfortunately divergent) expression for $\partial K_2/\partial x$

$$\frac{\partial K_2}{\partial x} = \frac{A_\nu}{2\pi}\int_0^\infty R^2\left(\frac{\delta R}{2}\right)^{2/n} J_0(rR)\,\mathrm{d}R$$

$$= \frac{A_\nu}{2\pi r^3}\left(\frac{\delta}{2r}\right)^{2/n}\int_0^\infty u^{2+2/n} J_0(u)\,\mathrm{d}u$$

Although the integral is divergent, we are tempted to assume that in fact

$$\partial K_2/\partial x = (C/r^3)\ (\delta/2r)^{2/n} \tag{8A}$$

in which C is unknown.

C may be evaluated by computing the Fourier transform of $\partial K_2/\partial x$ and equating it to the integrand in Eq. (6A).

Using the transformation Eq. (7A) we obtain

$$\left(\frac{\partial K_2}{\partial x}\right)^* = 2\pi CR\left(\frac{\delta R}{2}\right)^{2/n}\int_0^\infty \frac{J_0(x)\,\mathrm{d}x}{x^{2+2/n}} \tag{9A}$$

We take the finite part of this integral

$$\int_0^\infty \frac{J_0(x)\,\mathrm{d}x}{x^{2+2/n}} = -\frac{n}{n+2}\int_0^\infty \frac{J_1(x)\,\mathrm{d}x}{x^{1+2/n}} \tag{10A}$$

The integral on the right-hand side is listed by Abramowitz and Stegun (item No. 11.4.16, p. 486)[13]

$$\int_0^\infty x^\mu J_1(x)\,\mathrm{d}x = \frac{2^\mu\Gamma(1+\mu/2)}{\Gamma(1-\mu/2)} \tag{11A}$$

Using Eqs. 10A and (11A) in Eq. (9A), we obtain

$$(\partial K_2/\partial x)^* = -(\pi CA_\nu/2^{1+2/n})R(\delta R/2)^{2/n}$$

Equating this result to the integrand in Eq. (6A), we conclude that

$$\partial K_2/\partial x = -(2\nu^2/\pi r^3)\ (\delta/r)^{2/n}$$

$$r \equiv (x^2+y^2)^{1/2}$$

This is integrated to obtain the final result:

$$K_2 = \frac{-2\nu^2\delta^{2/n}}{\pi}\int_{-\infty}^{x}\frac{\mathrm{d}u}{[(u^2+y^2)^{1/2}]^{3+(2/n)}}$$

The integration cannot be carried out explicitly, but K_2 can be expressed as the product of a function of $|y|$ and a function of $x/|y|$, the latter of which is easily evaluated numerically

$$K_2 = -(2\nu^2/\pi y^2)\ (\delta/|y|)^{2/n}\cdot V(x/|y|)$$

$$V\left(\frac{x}{|y|}\right) \equiv \int_{-\infty}^{x/|y|}\frac{\mathrm{d}v}{(1+v^2)^{3/2+1/n}}$$

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